Recall the formal definition of a limit of a function $f : \mathbb{R} \to \mathbb{R}$. We say that $\lim_{x \to c} f(x) = L$ if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[0 < |x - c| < \delta \implies |f(x) - L| < \epsilon]$$

Problem 1
Let $f(x) = 3x - 4$. Let $\epsilon = \frac{1}{100}$. Find a $\delta > 0$ so that for all $x \in \mathbb{R}$,

$$0 < |x - 2| < \delta \implies |f(x) - 2| < \frac{1}{100}$$

Problem 2
Show that $\lim_{x \to 0} \frac{1}{x} \neq 100$ by proving the negation:

$$(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R}) \left[0 < |x| < \delta \wedge \left| \frac{1}{x} - 100 \right| \geq \epsilon \right]$$

Hint: since we are allowed to choose ϵ and x, it suffices to let $\epsilon = 1$ and only consider positive values of x.

Problem 3
For each of the following problems you may draw a graph to support your reasoning instead of giving a full proof.[a]

1. Define $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. Does $\lim_{x \to 0} h(x)$ exist?
2. Define $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. Does $\lim_{x \to 0} h(x)$ exist?
3. Define $h : (-1, 1) \setminus \{0\} \to \mathbb{R}$, $h(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. Does $\lim_{x \to 0} h(x)$ exist?

[a]Don't do this on your assignment.

Problem 4
Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\lim_{x\to\infty} f(x) = L.$$

1. Write down the definition of
$$\lim_{x\to\infty} f(x) = L.$$
2. Write down the definition of
$$\lim_{y\to 0^+} f\left(\frac{1}{y}\right) = L.$$
3. Show that
$$\lim_{y\to 0^+} f\left(\frac{1}{y}\right) = L.$$

Problem 5
Recall that if f, g are defined in some interval around $c \in \mathbb{R}$, and

$$\lim_{x\to c} f(x) = M \quad \text{and} \quad \lim_{x\to c} g(x) = N,$$

then

$$\lim_{x\to c}[f(x) + g(x)] = M + N \quad \text{and} \quad \lim_{x\to c}[f(x)g(x)] = MN.$$

This problem shows why it is necessary for $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ to exist in the above.

1. Give an example of an f, g defined in an interval around $c \in \mathbb{R}$ such that $\lim\limits_{x\to c}[f(x) + g(x)]$ exists but $\lim\limits_{x\to c} f(x)$ or $\lim\limits_{x\to c} g(x)$ don't exist.
2. Give an example of an f, g defined in an interval around $c \in \mathbb{R}$ such that $\lim\limits_{x\to c}[f(x)g(x)]$ exists but $\lim\limits_{x\to c} f(x)$ or $\lim\limits_{x\to c} g(x)$ don't exist.

Problem 6
Use squeeze theorem in the following questions.

1. Write down the statement of squeeze theorem.
2. Define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} x & x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$
Show that $\lim\limits_{x\to 0} f(x) = 0$.
3. Show that for any function $f : \mathbb{R} \to \mathbb{R}$,
$$\lim_{x\to c} |f(x)| = 0$$
if and only if
$$\lim_{x\to c} f(x) = 0.$$